General vector basis function solution of Maxwell's equations

Dipankar Sarkar and N. J. Halas
Department of Electrical and Computer Engineering and Center for Nanoscale Science and Technology, Rice University, P.O. Box 1892, Houston, Texas 77251

(Received 20 February 1997)

A general method for solving Maxwell's equations exactly, based on expansion of the solution in a complete set of vector basis functions, is developed. These vector eigenfunctions are derived from the complete set of separable solutions to the scalar Helmholtz equation and are shown to form a complete set. The method is applicable to a variety of problems, including the study of near and far field electromagnetic scattering from particles with arbitrary shapes, for particles whose characteristic length scale $d \approx \lambda$, the wavelength of the incident electromagnetic wave. [S1063-651X(97)06907-9]

PACS number(s): 03.40.Kf, 03.50.De, 02.60.Lj

I. INTRODUCTION

The subject of obtaining the solution of Maxwell's equations in a boundary value problem has kept scientists occupied for more than 100 years [1]. Many contemporary problems of interest involve our knowledge of the exact solution to these equations. However, the complexity of these equations has defied analytic solutions for all but the simplest cases.

In a scattering problem, for example, there are two lengths involved: d, the physical size of the scattering object and λ, the wavelength of electromagnetic waves in question. When $d \gg \lambda$, called the geometrical optics limit, the problem is readily solved. When $d \ll \lambda$, also called the quasistatic limit or nano-optics, the solution is obtained by solving Laplace's equation. The problem is far more complicated when $d \approx \lambda$. It is in this regime that one is required to solve Maxwell's equations exactly. Although the general theoretical aspects of Maxwell's equations are fairly well understood at the moment [1,2], there are still some formidable mathematical or computational problems to overcome. Whereas the problem of acoustic (scalar fields) diffraction has been solved in a number of cases such as the strip, elliptic cylinder, hyperbolic cylinder, wedge, prolate and oblate spheroids, etc. [3], the corresponding solution for the time-varying vector fields is still by and large unconquered. Often the solution is worked out only in the quasistatic approximation by solving Laplace's equation (homogeneous case) [4,5].

In the absence of sources, Maxwell's equations, which are coupled partial differential equations, have been solved analytically, only for a few isolated special cases. The simplest canonical case of the scattering of plane waves reflecting or refracting at a planar interface [6] is commonly known as Fresnel's equations. The most celebrated analytic solution to the scattering problem is that of Mie scattering, the scattering of electromagnetic waves from a dielectric sphere [7], originally due to Mie (1908). Debye was able to formulate the same problem in terms of a pair of coupled scalar functions [8], in which the electric and magnetic fields were expressed. The method was suitable for boundary value problems with spherical boundaries only. Hansen [9–11] developed a technique for addressing the problem of radiation from antennas using a special type of transformation. The subject was developed further by Stratton [12], who solved the problem of Mie scattering using a set of vector functions derived from the solutions of the scalar Helmholtz equation. Subsequently, Aden and Kerker [13] used the formalism to solve the problem of electromagnetic scattering from two concentric spheres. Although Stratton [12] had made considerable contributions in the area of basis function expansion, the method of solution was still considered as just another ''more elegant'' way to solve the Mie problem, analogous to the approach of Debye [8]. The crucial idea that was missing at this stage was the notion of mathematical completeness of basis function expansions. Furthermore, the algebraic difficulties one had to overcome restricted the Stratton approach to the solution of the spherical scatterer problem with incident light approaching along the z- axis only. Progress in this subject was negligible beyond this stage and the problem is reproduced in its original form even in contemporary textbooks [14,15]. The problem of diffraction of a plane wave at normal incidence on a circular cylinder was originally solved by Rayleigh and since then his solution has been generalized and extended to plane waves at oblique incidence [16]. The method of basis function expansion was used in this problem as well. Similarly, the problems of electromagnetic and acoustic scattering from a semi-infinite cone [17] and a semi-infinite body of revolution [18] were obtained by the method of basis function expansion. For the special case of the paraboloid, the solutions were ''exact.'' Some other notable geometries for which the exact solution of electromagnetic scattering has been obtained in three dimensions include the prolate and oblate ellipsoids. Certain two dimensional cases, such as the circular, parabolic, and elliptic cylinders, are also amenable for exact solutions [3]. However, for arbitrary geometrical shapes one has to resort to solving the Maxwell equations by finite element numerical techniques. The generality of these approaches was not apparently proved beyond the specific geometries these earlier researchers were interested in.

In the mid 1960s there began a growing interest in the study of Maxwell's equations as a purely mathematical problem [1,2]. For example, it was found that any electromagnetic response within a perfectly conducting cavity (resonator) could be expressed in a complete set of functions [1,2].

In this paper we discuss a general method for solving Maxwell's equations in a boundary value problem with linear, isotropic, homogeneous, and time-invariant media. We introduce the diffraction equation in the next section. Next we prove the existence of a complete set of eigenfunctions of the diffraction operator. This is followed by deriving the eigenfunctions in a spherical-polar coordinate system and the expansion coefficients for an arbitrary plane wave in the complete set of eigenfunctions. We then demonstrate numerical convergence aspects of these eigenfunction expansions. Finally, we apply the technique of vector basis function expansion to the problem of scattering of electromagnetic waves from an elongated scatterer.

II. THE MAXWELL EQUATIONS

For the interesting subclass of problems that are related to the scattering of electromagnetic radiation by dielectric and/or magnetic boundaries, Maxwell's equations assume a symmetrical form. Within the approximation of a linear, time-invariant, homogeneous, and isotropic medium without any external sources, Maxwell's equations are, assuming sinusoidal time dependence of the external fields, $e^{-i\omega t}$,

$$\nabla\cdot\mathbf{E}(\mathbf{r},\omega)=0, \tag{1}$$

$$\nabla\times\mathbf{E}(\mathbf{r},\omega)=i\omega\mu\mathbf{H}(\mathbf{r},\omega), \tag{2}$$

$$\nabla\cdot\mathbf{B}(\mathbf{r},\omega)=0, \tag{3}$$

$$\nabla\times\mathbf{H}(\mathbf{r},\omega)=\sigma\mathbf{E}(\mathbf{r},\omega)-i\omega\epsilon\mathbf{E}(\mathbf{r},\omega). \tag{4}$$

Here we have assumed the possibility of an induced current density in the medium equal to $\sigma\mathbf{E}$, where σ is the conductivity of the medium under consideration. Thus we obtain a set of coupled partial differential equations in the spatial coordinates alone, with the angular frequency of the excitation as a parameter.

Using the facility of complex dielectric functions, $\hat{\epsilon}=\epsilon+i\sigma/\omega$, we obtain the vector Helmholtz equations for the electric and magnetic field vectors:

$$\nabla\times\nabla\times\mathbf{E}-\omega^2\mu\hat{\epsilon}\mathbf{E}=0, \tag{5}$$

$$\nabla\times\nabla\times\mathbf{H}-\omega^2\mu\hat{\epsilon}\mathbf{H}=0. \tag{6}$$

These equations are totally symmetrical and decoupled in the field variables. It is the solution to the vector Helmholtz equations for specified boundary and radiation conditions that describes the scattering of electromagnetic waves. However, these equations are vector partial differential equations which are sufficiently difficult to solve in general.

The theory of solving the scalar Helmholtz equation, also known as the wave equation, is a very well developed subject [19–21]. This equation describes the propagation of scalar waves, such as acoustic waves, in a medium.

$$\nabla^2\psi(\mathbf{r},t)+k^2\psi(\mathbf{r},t)=0. \tag{7}$$

When expressed in certain orthogonal curvilinear coordinate systems, such as the Cartesian coordinate system or the spherical-polar coordinate system, under assumption of sinusoidal time dependence, the solution $\psi(\mathbf{r},t)$ can be obtained by the technique of separation of variables. The separation procedure reduces the partial differential equation to several ordinary differential equations. The separated equations can often be cast in the form of the well known Sturm-Liouville eigenvalue problem for second order ordinary differential equations, so that the solution space is guaranteed to be complete for the scalar Helmholtz equation.

For the moment assume that we have obtained a complete set of scalar functions $\{\psi\}$ that are solutions to the scalar wave equation [Eq. (7)]. We introduce the diffraction equation

$$\nabla(\nabla\cdot\mathbf{G})-\nabla\times\nabla\times\mathbf{G}+k^2\mathbf{G}=0, \tag{8}$$

where $k^2=\omega^2\mu\hat{\epsilon}$. The diffraction equation is satisfied by the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ that satisfies the vector Helmholtz equations since the ''extra'' term $\nabla\cdot\mathbf{G}$ will be zero for fields that have zero divergence. So, Eq. (8) is consistent with Maxwell's equations for electromagnetic waves. We define a vector function that is obtained by taking the gradient of the scalar function ψ:

$$\mathbf{L}=\nabla\psi. \tag{9}$$

$\mathbf{L}$ satisfies the diffraction equation if ψ is a solution to Eq. (7):

$$\begin{aligned}\nabla(\nabla\cdot\mathbf{L})&-\nabla\times\nabla\times\mathbf{L}+k^2\mathbf{L}\\&=\nabla(\nabla\cdot\nabla\psi)-\nabla\times\nabla\times\nabla\psi+k^2\nabla\psi\\&=\nabla(\nabla^2\psi+k^2\psi)\quad(\text{since }\nabla\times\nabla\psi=0)\\&=0\quad(\text{since }\nabla^2\psi+k^2\psi=0).\end{aligned} \tag{10}$$

Now let us assume that there exists a vector function $\mathbf{M}$, with zero divergence, $\nabla\cdot\mathbf{M}=0$, that is a solution to the diffraction equation [Eq. (8)]. Let us consider the vector $\mathbf{N}=(1/k)\nabla\times\mathbf{M}$. Assuming k is a constant (homogenous medium), we obtain

$$\begin{aligned}&\nabla(\nabla\cdot\mathbf{N})-\nabla\times\nabla\times\mathbf{N}+k^2\mathbf{N}\\&\quad=\frac{1}{k}[\nabla(\nabla\cdot\nabla\times\mathbf{M})-\nabla\times\nabla\times\nabla\times\mathbf{M}+k^2\nabla\times\mathbf{M}]\\&\quad=\nabla\times[-\nabla\times(\nabla\times\mathbf{M})+k^2\mathbf{M}]\quad[\text{since }\nabla\cdot(\nabla\times\mathbf{M})=0]\\&\quad=\nabla\times[\nabla(\nabla\cdot\mathbf{M})-\nabla\times(\nabla\times\mathbf{M})+k^2\mathbf{M}]\\&\qquad(\text{since }\nabla\cdot\mathbf{M}=0)\\&\quad=0\quad[\text{since }\mathbf{M}\text{ satisfies Eq. (8)}].\end{aligned}$$

Thus we show that $\mathbf{N}$, as defined above, also satisfies the diffraction equation. Also $\nabla\cdot\mathbf{N}=0$, since divergence of curl is zero. So, if we had started with postulating the existence of a vector function $\mathbf{N}$ with zero divergence, we could show that there exists $\mathbf{M}$ with zero divergence. Thus from symmetry arguments alone, we could write $\mathbf{M}=(1/k)\nabla\times\mathbf{N}$. Specifically, since $\mathbf{N}=(1/k)\nabla\times\mathbf{M}$, since k is a constant (by assumption of homogeneity of the medium),

$$\nabla \times \mathbf{N} = \frac{1}{k} \nabla \times \nabla \times \mathbf{M},$$

$$\nabla \times \mathbf{N} = \frac{1}{k} k^2 \mathbf{M} \quad [\text{follows from Eq. (8)}].$$

Thus $\mathbf{M} = (1/k) \nabla \times \mathbf{N}$. Clearly, **M** and **N** are distinct from **L**, since the latter has nonzero divergence in general. So **L** must be linearly independent of $\{\mathbf{M},\mathbf{N}\}$. It is up to us to create a vector function **M** (or equivalently **N**) from the given scalar function ψ, such that it will have zero divergence and will satisfy the diffraction equation. The point to note is that we could arrive at more than one set of functions $\{\mathbf{M},\mathbf{N}\}$, and it is the relative algebraic convenience that will dictate the choice of a particular set.

As a concrete example, let us consider $\mathbf{M} = \nabla \times \mathbf{a}\psi$, where ψ is the given scalar solution and **a** is an arbitrary constant vector. The divergence condition is satisfied, since divergence of curl is identically zero: $\nabla \cdot \mathbf{M} = \nabla \cdot (\nabla \times \mathbf{a}\psi) = 0$. Using the operator identity (the author has verified this ''identity'' for the spherical-polar coordinate sytem and the cylindrical coordinate system in addition to the rectangular coordinate system), $\nabla(\nabla \cdot \mathbf{M}) - \nabla \times \nabla \times \mathbf{M} = \nabla^2 \mathbf{M}$, we can write:

$$\begin{aligned}
&\nabla(\nabla \cdot \mathbf{M}) - \nabla \times \nabla \times \mathbf{M} + k^2 \mathbf{M} \\
&\quad = \nabla^2 \mathbf{M} + k^2 \mathbf{M} \\
&\quad = \nabla^2 (\nabla \times \mathbf{a}\psi) + k^2 (\nabla \times \mathbf{a}\psi) \\
&\quad = \nabla^2 [\nabla \psi \times \mathbf{a} + \psi (\nabla \times \mathbf{a})] + k^2 [\nabla \psi \times \mathbf{a} + \psi (\nabla \times \mathbf{a})] \\
&\quad = \nabla^2 (\nabla \psi \times \mathbf{a}) + k^2 (\nabla \psi \times \mathbf{a}) \quad (\text{since} \nabla \times \mathbf{a} = 0) \\
&\quad = (\nabla^2 (\nabla \psi) + k^2 \nabla \psi) \times \mathbf{a} \quad (\text{since } \nabla^2 \text{ does not act on } \mathbf{a}) \\
&\quad = [\nabla (\nabla^2 \psi + k^2 \psi)] \times \mathbf{a} = 0 \quad (\text{since}) \psi \text{ satisfies Eq. (7)}.
\end{aligned}$$

Also, $\mathbf{M} = \nabla \times \mathbf{a}\psi = \nabla \psi \times \mathbf{a} = \mathbf{L} \times \mathbf{a}$. So, $\mathbf{M} \cdot \mathbf{L} = 0$, i.e., **L** and **M** are orthogonal. Thus, given a countably infinite set of particular solutions to Eq. (7), $\{\psi_n\}$, that are finite, continuous, single valued, and with continuous partial derivatives; associated with each ψ_n one can obtain a triplet of mutually noncoplanar vector solutions $\{\mathbf{L}_n, \mathbf{M}_n, \mathbf{N}_n\}$, satisfying Eq. (8). Presumably, any arbitrary solution of the diffraction equation can be expressed as a linear combination of these vector functions. However, the existence of a generalized Fourier series expansion supposes that the set $\{\mathbf{L}_n, \mathbf{M}_n, \mathbf{N}_n\}$ forms a complete set.

Proving the completeness of the set $\{\mathbf{L}_n, \mathbf{M}_n, \mathbf{N}_n\}$ is a two step process. First, one has to prove the existence of a complete set of functions for the diffraction equation. Next, one has to show that the $\{\mathbf{L}_n, \mathbf{M}_n, \mathbf{N}_n\}$ set can indeed span the solution space of the diffraction equation, or equivalently, the vector Helmholtz equations. If the labeling index n is not countable, then an arbitrary solution could be expressed as a generalized Fourier integral of the basis functions with respect to the labeling parameter.

III. COMPLETENESS OF VECTOR EIGENFUNCTIONS

Our goal is to represent the solution of electromagnetic scattering problems in a series of vector eigenfunctions of the diffraction operator: to do so, such a set must be complete (in a mathematical sense). In other words, the set of functions under consideration must form a basis. It can be shown that the solution space of the diffraction operator cannot be spanned by any finite set of functions. Assume momentarily that we have obtained only the $\mathbf{L}_n$ functions from the scalar solution, and that we have no knowledge about the existence of the $\mathbf{M}_n$ or $\mathbf{N}_n$ functions. So we have a countably infinite set of functions that satisfy the diffraction operator. But it is easy to prove that such a set does not form a basis. In other words, there are elements in the solution space that are independent of the $\mathbf{L}_n$ functions. For example, the $\mathbf{M}_n$ functions defined as $\mathbf{L}_n \times \mathbf{a}$ are clearly orthogonal to the $\mathbf{L}_n$ functions. Naturally, the same concerns are valid even with knowledge of the larger set containing all three types of functions. We do need to address the question of whether or not the set $\{\mathbf{L}_n, \mathbf{M}_n, \mathbf{N}_n\}$ is a complete set.

It is a well established fact that the set of all plane wave solutions forms a complete set in the solution space of the vector Helmholtz operator. Physically this implies that any scattered wave can be constructed by linear addition or superposition of plane waves. The diffraction operator is somewhat more general than the vector Helmholtz operator in the sense that it is satisfied by ''generalized plane waves'' which could have nonzero divergence. The vector Helmholtz equation, which follows directly from Maxwell's equations in a homogeneous and isotropic medium, is satisfied only by the zero-divergence solutions. Within a nonisotropic medium in which momentum transfer can occur, such as in a crystal, the **E** need not be perpendicular to **k**, and the zero-divergence solutions cannot describe such a wave.

Although the set of all generalized plane waves spans the solution space, there is a fundamental difficulty with such a set, arising from the fact that such a set is uncountable. In other words, the label(s) to identify the individual elements of the set are in this case continuous variables (being the value of the propagation vector, and its direction cosines, and the direction cosines of the electric field). Although such sets are not easily amenable for construction of general scattered wave solutions, a countable basis can be used to construct solutions in a straightforward manner.

To begin, the solution space of the diffraction operator is linear, has an inner product and a metric. The space is complete since there exists the plane wave set which we know is a complete set. With these properties satisfied, the set of all solutions of the diffraction operator forms a Hilbert space. The inner product can be defined, as will be seen when considering the solutions in a particular coordinate system, such as the spherical polar coordinate system.

It can be shown that if $\{\mathbf{e}_i\}$ is an orthonormal set in a Hilbert space H, and if $\mathbf{x}$ is any vector in H, then the set $S = \{\mathbf{e}_i : \langle \mathbf{x} | \mathbf{e}_i \rangle \neq 0\}$ is either empty or countable [22]. The importance of this theorem is that this guarantees a countable basis if it exists. It can also be shown that every nonzero Hilbert space contains a complete orthonormal set. This follows from fundamental axioms of set theory embodied in

Zorn's lemma [23]. Thus a Hilbert space contains a countable basis.

A. Completeness of the L, M, N set

We have already seen that we can arrive at a set $\{\mathbf{L}_n, \mathbf{M}_n, \mathbf{N}_n\}$ of vector eigenfunctions that are mutually noncoplanar. In general $\nabla\psi_n$ and $\nabla\psi_{n'}$ are different functions of position coordinates. Similarly, it can be argued that the set of vector eigenfunctions is such that at any given point in space, they are not all pointing in the same direction. In mathematical language, the set is linearly independent. The set also has a countably infinite number of elements in it. The scalar functions $\{\psi_n\}$ are continuous with continuous partial derivatives up to at least second order. This implies that the derived vector functions are continuous as well.

Assume that we are given an arbitrary solution to the diffraction equation, $\mathbf{x}$. By saying that we are given the solution, we mean that its value has been specified at a given set of points, $S=\{\mathbf{x}_1, \mathbf{x}_2, \ldots\}$. For the moment let us assume that this set S is a countable set, as we have indicated by subscripting with natural numbers. For example, if the value has been specified at all coordinates (x,y,z) where $x,y,z \in \{p/q : p,q \in I\}$, then the set can be shown to be countable. Thus the specified set of points can be put in one-to-one correspondence with an index set, such as the natural numbers.

Now if we pick the first specified point where the function is defined, we obtain a vector of a certain magnitude which points along some specified direction. We can immediately pick the triplet of solutions $\{\mathbf{L}_1, \mathbf{M}_1, \mathbf{N}_1\}$, and by virtue of their linear independence, we can choose suitable coefficients so that the sum $a_1\mathbf{L}_1 + b_1\mathbf{M}_1 + c_1\mathbf{N}_1 = \mathbf{x}_1$, where the coefficients a_1, b_1, c_1 are complex numbers. Since the left hand side is a linear combination of solutions to the diffraction equation, we have a valid solution that is equal to the specified solution $\mathbf{x}$ at one point. In general the diffraction operator will propagate the linear combination $a_1\mathbf{L}_1 + b_1\mathbf{M}_1 + c_1\mathbf{N}_1$ so that it will be different from the specified solution at other points (if it does not, then of course we have obtained the desired expansion, and we can stop the process here). So assume that the linear combination just obtained deviates from the solution at point 2, $\mathbf{x}_2$. Now we can pick our second triplet $\{\mathbf{L}_2, \mathbf{M}_2, \mathbf{N}_2\}$, and obtain a ''correction'' to the original solution so that it matches at both the points. Essentially, we are solving a system of six equations in six unknowns to satisfy the match at the two specified points. Now it is easy to see that we can continue this process to ''match'' the specified function in an infinite series of the set of vector eigenfunctions to any arbitrary precision. By virtue of continuity of these functions, their linear combinations are also continuous for any finite number of terms. The difference between the arbitrary solution and the series just obtained can in principle be made as small as we wish; i.e., it forms a Cauchy sequence. We note that the set of expansion coefficients so obtained need *not* be unique. It does depend upon the order in which we picked the vector eigenfunctions to satisfy the conditions for a match.

The validity of the statement of convergence of arbitrary Cauchy sequences follows from general considerations of a more restricted space of square-integrable functions, L^2, with a semimetric. We know that the scattered solutions have to satisfy the radiation conditions, i.e., they have to vanish at infinity in a square-integrable sense. This follows from the finite energy content in any scattered wave from finite objects. The **L**, **M**, and **N** functions satisfy such conditions. It can be shown (the Reisz-Fischer theorem) that every Cauchy sequence in the semimetric space L^2 converges to a function in L^2, or that it is complete [24].

Although the set of functions $\{\mathbf{L}_n, \mathbf{M}_n, \mathbf{N}_n\}$ is not entirely orthogonal, it is a linearly independent set. This allows us to invoke the process known as Gram-Schmidt orthonormalization to obtain a complete orthonormal set from a given set of linearly independent vectors [22]. In practice, this is not always necessary.

B. Zero-divergence solutions

Consider a solution $\mathbf{F}$ whose divergence is zero. Let us find an expansion of $\mathbf{F}$ in terms of the basis $\{\mathbf{L}_n, \mathbf{M}_n, \mathbf{N}_n\}$, so that

$$\mathbf{F} = \sum_n \{a_n\mathbf{M}_n + b_n\mathbf{N}_n + c_n\mathbf{L}_n\}. \tag{11}$$

Taking the divergence of both sides of the above equation, we find

$$\nabla\cdot\mathbf{F} = \sum_n \{a_n\nabla\cdot\mathbf{M}_n + b_n\nabla\cdot\mathbf{N}_n + c_n\nabla\cdot\mathbf{L}_n\}$$

or

$$0 = \sum_n \{c_n\nabla\cdot\mathbf{L}_n\}. \tag{12}$$

Since this must hold true at all points, we conclude that all the c_n must be zero. In other words, a zero-divergence solution can be expressed only in terms of the **M** and **N** functions.

IV. THE L, M, N BASIS

The success of the method of vector basis function expansion relies on our ability to express, conveniently, the incident radiation in terms of these basis functions. Although the completeness of this basis set guarantees the existence of unique coefficients for the expansion of an incident plane wave, in practice this involves some nontrivial algebra. The goal is to obtain explicit expansion coefficients for a general plane wave in these basis eigenfunctions. Linearity of Maxwell's equations implies that any excitation, nonplanar or even nonperiodic, could in principle be solved from the known solution to plane wave excitations by use of Fourier space analysis.

In principle, the set of basis functions obtained in any general orthogonal coordinate system could be considered a valid set of eigenfunctions in which the solution could be expressed. However, strictly algebraic and numerical considerations favor the use of functions pertaining to the spherical-polar coordinate system, especially for systems with azimuthal symmetry. The extensive analytical foundation of spherical Bessel or Hankel functions as well as associated Legendre functions allows us to obtain the expansion coefficients in closed form expression.

The solutions to the scalar Helmholtz equation in spherical-polar coordinates are functions of the form

$$\psi_{mn}(r,\theta,\phi)\sim z_n(kr)P_n^m(\cos\,\theta)e^{im\phi}, \tag{13}$$

where $z_n(kr)$ represents either the spherical Bessel functions $j_n(kr)$ or the spherical Hankel functions of the first kind, $h_n^{(1)}(kr)$. The spherical Bessel functions are regular at the origin, whereas the spherical Hankel functions diverge at or near the origin. So a region including the origin can only feature the spherical Bessel functions in its expression for the field. A region not including the origin can have contributions from either of these functions. The labeling index $n\in\{0,1,2,\ldots\}$ and $m\in\{0,\pm1,\pm2,\ldots,\pm n\}$. The $P_n^m(\cos\,\theta)$ are the associated Legendre functions.

Analogous to the definitions Stratton [12] uses, we obtain explicit expressions for the **L**, **M**, **N** functions [25]:

$$\begin{aligned}\mathbf{L}_{mn}&=\boldsymbol{\nabla}\psi_{mn}\\&=k\left\{\frac{dz_n(kr)}{d(kr)}P_n^m(\cos\,\theta)e^{im\phi}\mathbf{e}_r+\frac{z_n(kr)}{kr}\frac{dP_n^m(\cos\,\theta)}{d\theta}\right.\\&\quad\left.\times e^{im\phi}\mathbf{e}_\theta+im\frac{z_n(kr)}{kr}\frac{P_n^m(\cos\,\theta)}{\sin\,\theta}e^{im\phi}\mathbf{e}_\phi\right\},\end{aligned} \tag{14}$$

$$\begin{aligned}\mathbf{M}_{mn}&=imz_n(kr)\frac{P_n^m(\cos\,\theta)}{\sin\,\theta}e^{im\phi}\mathbf{e}_\theta\\&\quad-z_n(kr)\frac{dP_n^m(\cos\,\theta)}{d\theta}e^{im\phi}\mathbf{e}_\phi,\end{aligned} \tag{15}$$

and

$$\begin{aligned}\mathbf{N}_{mn}&=n(n+1)\frac{z_n(kr)}{kr}P_n^m(\cos\,\theta)e^{im\phi}\mathbf{e}_r\\&\quad+\frac{1}{kr}\frac{drz_n(kr)}{dr}\frac{dP_n^m(\cos\,\theta)}{d\theta}e^{im\phi}\mathbf{e}_\theta\\&\quad+im\frac{1}{kr}\frac{drz_n(kr)}{dr}\frac{P_n^m(\cos\,\theta)}{\sin\,\theta}e^{im\phi}\mathbf{e}_\phi.\end{aligned} \tag{16}$$

V. PLANE WAVE EXPANSION IN L, M, N BASIS

The completeness of the set of functions **L**, **M**, and **N** assures us of the existence of a valid expansion series in this basis set for an arbitrary plane electromagnetic wave which satisfies the diffraction equation. Once a valid set of coefficients is obtained, we can offer an operational proof of the completeness theorem. If any plane wave can be represented in a convergent series of these functions, then it immediately implies that the set must be complete. This follows from the fact that any scattered solution to the vector Helmholtz equation can be constructed from a plane wave basis.

Consider an arbitrary plane wave:

$$\mathbf{F}=\mathbf{E}e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{17}$$

Here $\mathbf{E}=E_x\mathbf{e}_x+E_y\mathbf{e}_y+E_z\mathbf{e}_z$ is the (oscillating) electric field of the plane wave, $\mathbf{k}=(k,\alpha,\beta)$ is the propagation vector, and $\mathbf{r}=(r,\theta,\phi)$ is the position coordinate. To keep our derivation sufficiently general, we shall allow the orientation of **E** and **k** to be along arbitrary directions. In reality, for a plane electromagnetic wave in a homogeneous and isotropic medium, the **E** and **k** must be orthogonal, and so the divergence of **F** must vanish. We shall also allow E_x, E_y, E_z, and k to be complex numbers. This represents any general elliptically polarized wave propagating in a medium that can be either attenuating or amplifying.

The completeness of the set of functions validates the following representation:

$$\mathbf{e}_xe^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{mn}\{a_{mn}^x\mathbf{M}_{mn}+b_{mn}^x\mathbf{N}_{mn}+c_{mn}^x\mathbf{L}_{mn}\}, \tag{18}$$

$$\mathbf{e}_ye^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{mn}\{a_{mn}^y\mathbf{M}_{mn}+b_{mn}^y\mathbf{N}_{mn}+c_{mn}^y\mathbf{L}_{mn}\}, \tag{19}$$

$$\mathbf{e}_ze^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{mn}\{a_{mn}^z\mathbf{M}_{mn}+b_{mn}^z\mathbf{N}_{mn}+c_{mn}^z\mathbf{L}_{mn}\}. \tag{20}$$

Therefore we can write the general plane wave expansion as

$$\begin{aligned}\mathbf{E}e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{mn}\{&(E_xa_{mn}^x+E_ya_{mn}^y+E_za_{mn}^z)\mathbf{M}_{mn}\\&+(E_xb_{mn}^x+E_yb_{mn}^y+E_zb_{mn}^z)\mathbf{N}_{mn}\\&+(E_xc_{mn}^x+E_yc_{mn}^y+E_zc_{mn}^z)\mathbf{L}_{mn}.\end{aligned} \tag{21}$$

If **E** is represented by (E,θ_i,ϕ_i), in the spherical-polar coordinate system, then

$$E_x=E\,\sin\,\theta_i\,\cos\,\phi_i,\quad E_y=E\,\sin\,\theta_i\,\sin\,\phi_i,$$
$$E_z=E\,\cos\,\theta_i. \tag{22}$$

Therefore we need to obtain the nine coefficients $\{a_{mn}^x,a_{mn}^y,\ldots,c_{mn}^z\}$ which are functions of $\{n,m,\alpha,\beta\}$.

It can be shown that the exponential part of the expression for a plane wave can be expanded in the following series:

$$\begin{aligned}e^{i\mathbf{k}\cdot\mathbf{r}}=\sum_{s=0}^{\infty}&i^s(2s+1)j_s(kr)\\&\times\left\{\sum_{l=0}^{s}\frac{(s-l)!}{(s+l)!}P_s^l(\cos\,\alpha)e^{-il\beta}P_s^l(\cos\,\theta)e^{il\phi}\right.\\&\left.+\sum_{l=1}^{s}\frac{(s-l)!}{(s+l)!}P_s^l(\cos\,\alpha)e^{il\beta}P_s^l(\cos\,\theta)e^{-il\phi}\right\}.\end{aligned} \tag{23}$$

We denote $\mathbf{e}_ce^{i\mathbf{k}\cdot\mathbf{r}}\equiv\mathbf{E}_c$, where c denotes x, y, or z. We introduce the notation

$$\int_0^\infty dr \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \mathbf{E}_c \cdot \mathbf{X}_{mn}^* = \int \mathbf{E}_c \cdot \mathbf{X}_{mn}^* = \langle \mathbf{E}_c | \mathbf{X}_{mn} \rangle,$$

where $\mathbf{X}_{mn}$ represents $\mathbf{L}_{mn}$, $\mathbf{M}_{mn}$, or $\mathbf{N}_{mn}$. We can view $\langle \mathbf{E}_c | \mathbf{M}_{mn} \rangle$ as the definition of the *inner product* in the Hilbert space of all solutions to the diffraction equation.

From Eq. (20), we can calculate the coefficients $\{a_{mn}^z, a_{mn}^z, c_{mn}^z\}$ using certain orthogonality relations [25]. Obtaining the inner product of both sides of the equation with respect to $\mathbf{L}_{mn}$, $\mathbf{M}_{mn}$, and $\mathbf{N}_{mn}$, respectively, we obtain the following relations:

$$a_{mn}^z = \frac{\langle \mathbf{E}_z | \mathbf{M}_{mn} \rangle}{\langle \mathbf{M}_{mn} | \mathbf{M}_{mn} \rangle}, \tag{24}$$

$$b_{mn}^z = \frac{\langle \mathbf{E}_z | \mathbf{L}_{mn} \rangle \langle \mathbf{L}_{mn} | \mathbf{N}_{mn} \rangle - \langle \mathbf{E}_z | \mathbf{N}_{mn} \rangle \langle \mathbf{L}_{mn} | \mathbf{L}_{mn} \rangle}{\langle \mathbf{N}_{mn} | \mathbf{L}_{mn} \rangle \langle \mathbf{L}_{mn} | \mathbf{N}_{mn} \rangle - \langle \mathbf{N}_{mn} | \mathbf{N}_{mn} \rangle \langle \mathbf{L}_{mn} | \mathbf{L}_{mn} \rangle}, \tag{25}$$

$$c_{mn}^z = \frac{\langle \mathbf{E}_z | \mathbf{N}_{mn} \rangle \langle \mathbf{N}_{mn} | \mathbf{L}_{mn} \rangle - \langle \mathbf{E}_z | \mathbf{L}_{mn} \rangle \langle \mathbf{N}_{mn} | \mathbf{N}_{mn} \rangle}{\langle \mathbf{N}_{mn} | \mathbf{L}_{mn} \rangle \langle \mathbf{L}_{mn} | \mathbf{N}_{mn} \rangle - \langle \mathbf{N}_{mn} | \mathbf{N}_{mn} \rangle \langle \mathbf{L}_{mn} | \mathbf{L}_{mn} \rangle}. \tag{26}$$

By substituting the label z with x or y in Eqs. (24)–(26), we immediately obtain the corresponding formula for the remaining coefficients in terms of their inner products with the basis set functions.

The integrals represented by the various inner products in the expressions for the coefficients require a good deal of messy algebra and careful analysis for their evaluation [25]. The integrands involve products of spherical Bessel functions, associated Legendre functions, as well as their derivatives. The nontriviality of these integrations is amplified by the fact that one has to evaluate a double infinite summation (over s and l) to arrive at a closed form expression for these integrals. On substituting the expressions for these integrals back into Eqs. (24)–(26), we finally obtain the functional form for the coefficients (for $m \geqslant 0$):

$$\underset{(m\geqslant 0)}{a_{mn}^z(n,m,\alpha,\beta)} = i^{n+1} \frac{m(2n+1)}{n(n+1)} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\alpha) e^{-im\beta}, \tag{27}$$

$$\underset{(m\geqslant 0)}{b_{mn}^z(n,m,\alpha,\beta)} = \frac{i^{n+1}}{n(n+1)} \frac{(n-m)!}{(n+m)!} e^{-im\beta} \left[\begin{array}{c} n(n-m+1) P_{n+1}^m(\cos\alpha) \\ -(n+1)(n+m) P_{n-1}^m(\cos\alpha) \end{array} \right], \tag{28}$$

$$\underset{(m\geqslant 0)}{c_{mn}^z(n,m,\alpha,\beta)} = \frac{i^{n-1}}{k} \frac{(n-m)!}{(n+m)!} (2n+1) e^{-im\beta} \cos\alpha P_n^m(\cos\alpha). \tag{29}$$

We observe that when $\alpha = \pi/2$, so that $\mathbf{E}$ and $\mathbf{k}$ are perpendicular to each other as in a plane electromagnetic wave, all the c_{mn}^z coefficients vanish, since the divergence of such a field is zero. When $\alpha = 0$ or π all the a_{mn}^z coefficients vanish since $P_n^m(\pm 1) = 0$ for $m > 0$, and a_{mn}^z vanish for $m = 0$ because of the factor m in its expression. Similarly, the b_{mn}^z vanish for $\alpha = 0$ or π since $P_n^m(\pm 1) = 0$ for $m > 0$, and for $m = 0$ $P_{n'}(\pm 1)(-1)^{n'}$ for $m = 0$ so that the expression within the square brackets becomes for $m = 0$, $n(n+1)1 - (n+1)n = 0$. So when $\alpha = 0$ or π only the c_{mn}^z coefficients can be nonzero.

When $m < 0$, we can arrive at the corresponding coefficients by examining the changes that occur in the expressions for the individual inner product terms. It can be shown from considerations of parity of the associated Legendre functions [25] that the corresponding coefficients when $m < 0$ are given by

$$\underset{(m\geqslant 0)}{a_{-mn}^z(n,m,\alpha,\beta)} = (-1)^{m+1} e^{2im\beta} \frac{(n+m)!}{(n-m)!} a_{mn}^z(n,m,\alpha,\beta), \tag{30}$$

$$\underset{(m\geqslant 0)}{b_{-mn}^z(n,m,\alpha,\beta)} = (-1)^{m} e^{2im\beta} \frac{(n+m)!}{(n-m)!} b_{mn}^z(n,m,\alpha,\beta), \tag{31}$$

$$\underset{(m\geqslant 0)}{c_{-mn}^z(n,m,\alpha,\beta)} = (-1)^{m} e^{2im\beta} \frac{(n+m)!}{(n-m)!} c_{mn}^z(n,m,\alpha,\beta). \tag{32}$$

Similarly, the x coefficients are given as

$$\underset{(m\geqslant 0)}{a_{mn}^x(n,m,\alpha,\beta)} = i^{n+1} \frac{(2n+1)}{2n(n+1)} \frac{(n-m)!}{(n+m)!} \times \left[\begin{array}{r} (n+m)(n-m+1) P_n^{m-1}(\cos\alpha) e^{-i(m-1)\beta} \\ + P_n^{m+1}(\cos\alpha) e^{-i(m+1)\beta} \end{array} \right], \tag{33}$$

$$\underset{(m\geqslant 0)}{b^x_{mn}(n,m,\alpha,\beta)}=i^{n+1}\frac{1}{2n(n+1)}\frac{(n-m)!}{(n+m)!}\times\begin{bmatrix} (n+1)(n+m)(n+m-1)P_{n-1}^{m-1}(\cos\,\alpha)e^{-i(m-1)\beta} \\ -(n+1)P_{n-1}^{m+1}(\cos\,\alpha)e^{-i(m+1)\beta} \\ +n(n-m+2)(n-m+1)P_{n+1}^{m-1}(\cos\,\alpha)e^{-i(m-1)\beta} \\ -nP_{n+1}^{m+1}(\cos\,\alpha)e^{-i(m+1)\beta}\end{bmatrix}, \tag{34}$$

$$\underset{(m\geqslant 0)}{c^x_{mn}(n,m,\alpha,\beta)}=\frac{i^{n+1}}{2k}\frac{(n-m)!}{(n+m)!}\times\begin{bmatrix} (n+m)(n+m-1)P_{n-1}^{m-1}(\cos\,\alpha)e^{-i(m-1)\beta} \\ -P_{n-1}^{m+1}(\cos\,\alpha)e^{-i(m+1)\beta} \\ -(n-m+2)(n-m+1)P_{n+1}^{m-1}(\cos\,\alpha)e^{-i(m-1)\beta} \\ +P_{n+1}^{m+1}(\cos\,\alpha)e^{-i(m+1)\beta}\end{bmatrix}. \tag{35}$$

For $m<0$, the easiest way to obtain the coefficients are the following [25]:

$$a^x_{-mn}=(-1)^{m+1}\frac{(n+m)!}{(n-m)!}\begin{bmatrix} a^x_{mn} \text{ with } e^{-i(m\pm1)\beta} \text{ factors} \\ \text{changed to } e^{i(m\pm1)\beta}\end{bmatrix}, \tag{36}$$

$$b^x_{-mn}=(-1)^{m}\frac{(n+m)!}{(n-m)!}\begin{bmatrix} b^x_{mn} \text{ with } e^{-i(m\pm1)\beta} \text{factors} \\ \text{changed to } e^{i(m\pm1)\beta}\end{bmatrix}, \tag{37}$$

$$c^x_{-mn}=(-1)^{m}\frac{(n+m)!}{(n-m)!}\begin{bmatrix} c^x_{mn} \text{ with } e^{-i(m\pm1)\beta} \text{factors} \\ \text{changed to } e^{i(m\pm1)\beta}\end{bmatrix}. \tag{38}$$

The y-coefficients are obtained almost identically as compared with the x coefficients. We have

$$\underset{(m\geqslant 0)}{a^y_{mn}(n,m,\alpha,\beta)}=i^{n}\frac{(2n+1)}{2n(n+1)}\frac{(n-m)!}{(n+m)!}\times\begin{bmatrix} (n+m)(n-m+1)P_{n}^{m-1}(\cos\,\alpha)e^{-i(m-1)\beta} \\ -P_{n}^{m+1}(\cos\,\alpha)e^{-i(m+1)\beta}\end{bmatrix}, \tag{39}$$

$$\underset{(m\geqslant 0)}{b^y_{mn}(n,m,\alpha,\beta)}=i^{n}\frac{1}{2n(n+1)}\frac{(n-m)!}{(n+m)!}\times\begin{bmatrix} (n+1)(n+m)(n+m-1)P_{n-1}^{m-1}(\cos\,\alpha)e^{-i(m-1)\beta} \\ +(n+1)P_{n-1}^{m+1}(\cos\,\alpha)e^{-i(m+1)\beta} \\ +n(n-m+2)(n-m+1)P_{n+1}^{m-1}(\cos\,\alpha)e^{-i(m-1)\beta} \\ +nP_{n+1}^{m+1}(\cos\,\alpha)e^{-i(m+1)\beta}\end{bmatrix}, \tag{40}$$

$$\underset{(m\geqslant 0)}{c^y_{mn}(n,m,\alpha,\beta)}=\frac{i^{n}}{2k}\frac{(n-m)!}{(n+m)!}\times\begin{bmatrix} (n+m)(n+m-1)P_{n-1}^{m-1}(\cos\,\alpha)e^{-i(m-1)\beta} \\ +P_{n-1}^{m+1}(\cos\,\alpha)e^{-i(m+1)\beta} \\ -(n-m+2)(n-m+1)P_{n+1}^{m-1}(\cos\,\alpha)e^{-i(m-1)\beta} \\ -P_{n+1}^{m+1}(\cos\,\alpha)e^{-i(m+1)\beta}\end{bmatrix}. \tag{41}$$

Transformations similar to what was used for the x coefficients for $m<0$ are going to be valid for the y coefficients as well. Thus the y coefficients for negative m can be obtained as follows [note the extra (-1) factor as compared to the x transformations]:

$$a^y_{-mn}=(-1)^{m}\frac{(n+m)!}{(n-m)!}\begin{bmatrix} a^y_{mn} \text{ with } e^{-i(m\pm1)\beta} \text{factors} \\ \text{changed to } e^{i(m\pm1)\beta}\end{bmatrix}, \tag{42}$$

$$b^y_{-mn}=(-1)^{m+1}\frac{(n+m)!}{(n-m)!}\begin{bmatrix} b^y_{mn} \text{ with } e^{-i(m\pm1)\beta} \text{ factors} \\ \text{changed to } e^{i(m\pm1)\beta}\end{bmatrix}, \tag{43}$$

$$c^y_{-mn}=(-1)^{m+1}\frac{(n+m)!}{(n-m)!}\begin{bmatrix} c^y_{mn} \text{ with } e^{-i(m\pm1)\beta} \text{ factors} \\ \text{changed to } e^{i(m\pm1)\beta}\end{bmatrix}. \tag{44}$$

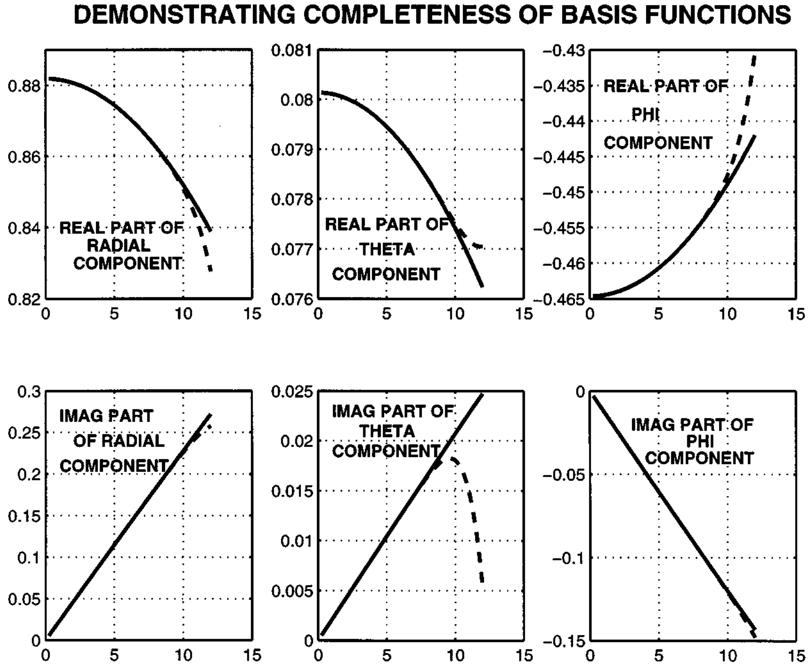

FIG. 1. Demonstrating completeness of basis function expansion. The expansion of an arbitrarily chosen wave in the **L**, **M**, and **N** functions. The solid lines show exact values. The broken lines are computed from a truncated series in the basis functions. The above calculations are done with 15 terms ($n=15$) for $kr\in[0,12]$, $\theta=37°$, $\phi=59°$, $\alpha=123°$, $\beta=83°$, $\theta_e=49°$, and $\phi_e=21°$. Convergence is very good for $|kr|\leqslant 0.75n$. The horizontal axes are in units of $|kr|$.

Numerical convergence of basis function expansion

It may be pointed out that the coefficients for the expansion of an arbitrary plane wave in terms of the **L, M, N** basis are not necessarily unique. Alternate sets of expansion coefficients can be obtained by rearranging the basis. This follows directly from the general theory of orthonormal bases in Hilbert spaces. Since one can expand any plane wave in this basis, as discussed earlier, one can obtain an operational proof of the completeness of this basis. It is also important to address the question of convergence as well. So numerical verifications are absolutely necessary to validate a certain basis set. To summarize the results, convergence of better than 1% is normally achieved by taking terms up to index n, where $n\sim 1.4|kr|$. This can provide us with the guideline on how many terms to include for a problem in which the geometry can be measured in units of λ, the wavelength of the electromagnetic field in question.

In Figs. 1 and 2 we show the expansion of arbitrary electromagnetic waves in the **L, M, N** basis. We have specified an arbitrarily directed wave vector **k** with direction cosines α and β, with the electric field oriented along some arbitrary direction specified by the angles θ_e and ϕ_e. The values of the exact expression for $\mathbf{E}e^{i\mathbf{k}\cdot\mathbf{r}}$ and the series expansion in the **L, M, N** basis are compared along some arbitrarily specified line directed along (θ,ϕ). We observe that the convergence is very good for $|kr|\leqslant 0.75n$ and this is fairly inde-

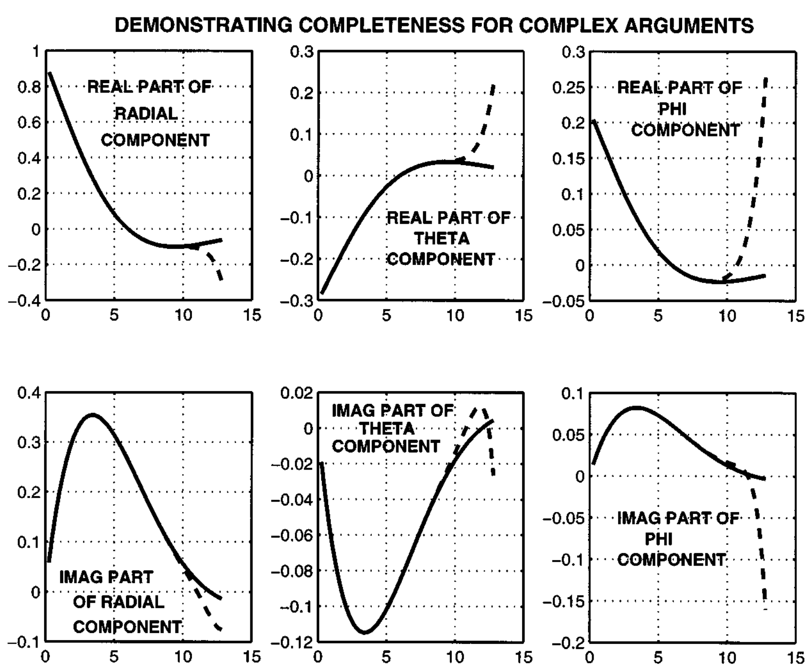

FIG. 2. Demonstrating completeness of basis function expansion when the specified wave vector is complex. The solid lines show exact values. The broken lines are computed from a truncated series in the basis functions. The above calculations are done with 15 terms ($n=15$) for $kr_{\max}=10+8i$, $\theta=35°$, $\phi=42°$, $\alpha=97°$, $\beta=82°$, $\theta_e=21°$, and $\phi_e=79°$. Convergence is very good for $|kr|\leqslant 0.75n$. The horizontal axes are in units of $|kr|$.

pendent of the choice of the specified directions for **E** and **k** as well as the path along which the comparison calculations are made. In Fig. 1 the wave vector is real whereas in Fig. 2 the wave vector is complex. Complex wave vectors correspond to an attenuating or amplifying medium. We therefore demonstrate that these functions can be used to expand an electromagnetic field in a general medium. Since the specified **E** and **k** are not orthogonal in general (as in the above two cases), it is therefore possible to expand waves that are more general than plane electromagnetic waves in a nonattenuating medium.

VI. SCATTERING FROM NONSPHERICAL OBJECTS

As an application of the method of vector basis function expansion, we discuss an ''exact'' method of solution for scattering from an object whose boundaries do not conform to the coordinate surfaces in a given coordinate system (the spherical-polar coordinate system in this case).

The scattering object is a ''capsule,'' composed of two hemispheres of radius R, separated by a cylinder of radius R and length L. The origin is chosen to coincide with the center of the lower hemisphere as indicated in Fig. 3. When $L\to 0$ the capsule degenerates to a sphere of radius R whose scattering solution can be obtained exactly. The axis of symmetry coincides with the z axis.

The eigenfunctions satisfy Maxwell's equations as well as the boundary conditions at infinity. The expansion coefficients of the scattered solution are determined by requiring that the boundary conditions on the surface of the ''capsule'' are satisfied at a finite number of chosen points. In principle, the larger the number of points we choose to specify, the more accurate will be the description of the boundary.

Thus there are two regions: inside the capsule surface where the dielectric function is ϵ_1, and outside the surface where the dielectric function is ϵ_2. We assume ϵ_2 to be a real function of the incident wave frequency. This validates the expansion of the incident electric field in the spherical Bessel function solutions. Since the origin is enclosed within the surface, the solution in region 1 will have the spherical Bessel function solutions only.

Consider an s-polarized incident wave of unit strength ($|\mathbf{E}_i|=1$) with the polarization oriented along the y-axis and **k** confined to the x-z plane ($\beta=0$):

$$\mathbf{E}_i=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\{a_{mn}\mathbf{m}_{emn}^{2j}+b_{mn}\mathbf{n}_{omn}^{2j}\}, \tag{45}$$

$$\mathbf{H}_i=-i\sqrt{\frac{\epsilon_2}{\mu_2}}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\{b_{mn}\mathbf{m}_{omn}^{2j}+a_{mn}\mathbf{n}_{emn}^{2j}\}, \tag{46}$$

where $a_{mn}=2a_{mn}^{y}$ and $b_{mn}=2ib_{mn}^{y}$, and the superscript 2 for the **m** and **n** functions refers to medium 2. The superscripts j and h refer to spherical Bessel functions or spherical Hankel functions of the first kind. Here

$$\frac{1}{2i}(\mathbf{M}_{mn}-\mathbf{M}_{mn}^{*})=\mathbf{m}_{omn}, \tag{47}$$

$$\frac{1}{2}(\mathbf{M}_{mn}+\mathbf{M}_{mn}^{*})=\mathbf{m}_{emn}, \tag{48}$$

$$\frac{1}{2i}(\mathbf{N}_{mn}-\mathbf{N}_{mn}^{*})=\mathbf{n}_{omn}, \tag{49}$$

$$\frac{1}{2}(\mathbf{N}_{mn}+\mathbf{N}_{mn}^{*})=\mathbf{n}_{emn}. \tag{50}$$

We assume the scattered electromagnetic fields in the two regions to have the following forms (expanding in the $m\geqslant 0$ terms only) :

$$\mathbf{E}_1=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\{a_{mn}^{1j}\mathbf{m}_{emn}^{1j}+b_{mn}^{1j}\mathbf{n}_{omn}^{1j}\}, \tag{51}$$

$$\mathbf{H}_1=-i\sqrt{\frac{\epsilon_1}{\mu_1}}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\{b_{mn}^{1j}\mathbf{m}_{omn}^{1j}+a_{mn}^{1j}\mathbf{n}_{emn}^{1j}\}, \tag{52}$$

$$\mathbf{E}_2=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\{a_{mn}^{2h}\mathbf{m}_{emn}^{2h}+b_{mn}^{2h}\mathbf{n}_{omn}^{2h}\}, \tag{53}$$

$$\mathbf{H}_2=-i\sqrt{\frac{\epsilon_2}{\mu_2}}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\{b_{mn}^{2h}\mathbf{m}_{omn}^{2h}+a_{mn}^{2h}\mathbf{n}_{emn}^{2h}\} \tag{54}$$

For each point on the surface, we can write down six equations corresponding to the boundary conditions. Not all of them are linearly independent. In fact when $L=0$, corresponding to a sphere, the equations of continuity of the tangential fields of **E** and **H** alone will yield a linearly independent set:

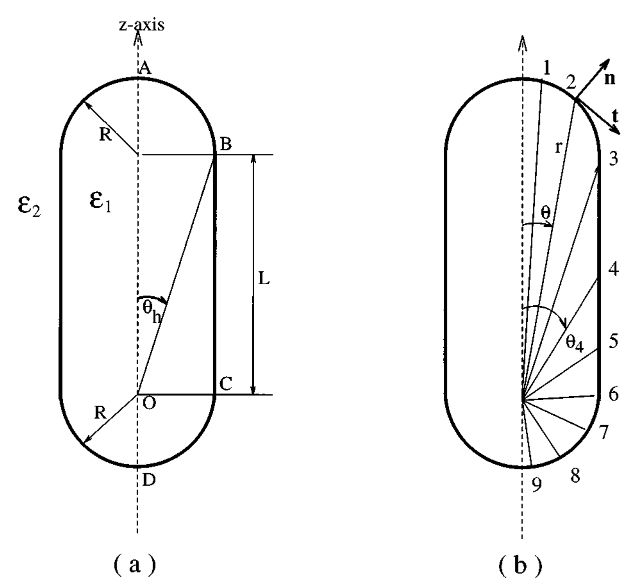

FIG. 3. (a) Geometry of ''capsule'' shaped scattering object. Two hemispheres of radius R are attached to the end of a cylinder of length L. Letting $L\to 0$, the capsule degenerates to a sphere. (b) Approximating the azimuthally symmetrical surface by means of the discrete set of angles θ_i. r as a function of θ is specified by a piecewise continuous function. **n** indicates the normal and **t** the tangent orthogonal to $\mathbf{e}_\phi$.

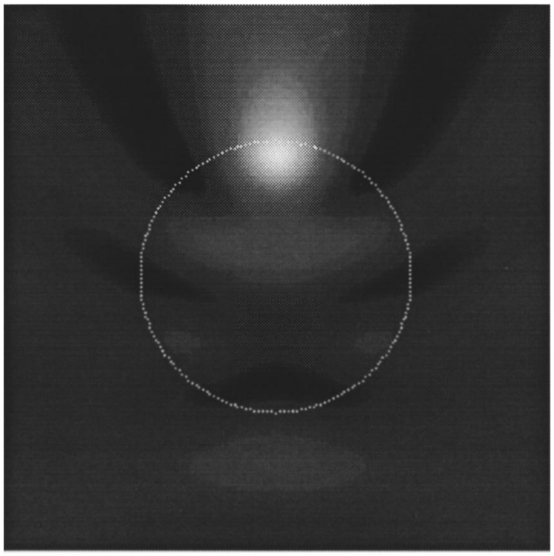

FIG. 4. Near field electromagnetic scattering from a spherical scatterer. $R=0.5\lambda$ and $L=0$. The incident light approaches along $\theta_i=0^{\mathbf{o}}$. Polarization is directed perpendicular to the plane of the figure.

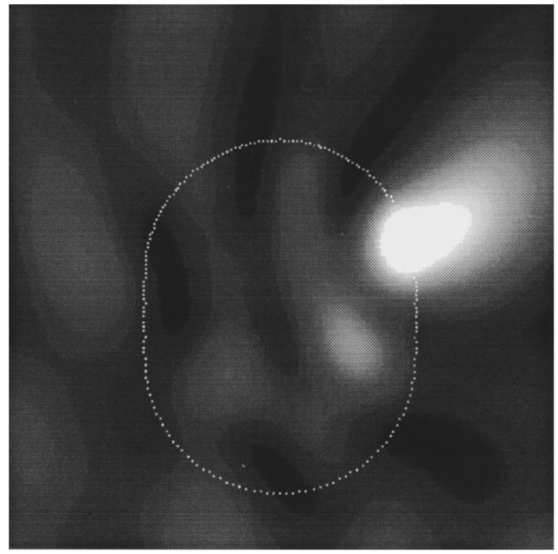

FIG. 5. Near field electromagnetic scattering from a ''capsule'' scatterer. $R=0.5\lambda$ and $L=0.3\lambda$. The incident light approaches along $\theta_i=60^{\mathbf{o}}$. Polarization is directed perpendicular to the plane of the figure.

$$\mathbf{E}_1\cdot\mathbf{t}_1=\mathbf{E}_2\cdot\mathbf{t}_1+\mathbf{E}_i\cdot\mathbf{t}_1, \tag{55}$$

$$\mathbf{E}_1\cdot\mathbf{t}_2=\mathbf{E}_2\cdot\mathbf{t}_2+\mathbf{E}_i\cdot\mathbf{t}_2, \tag{56}$$

$$\mathbf{H}_1\cdot\mathbf{t}_1=\mathbf{H}_2\cdot\mathbf{t}_1+\mathbf{H}_i\cdot\mathbf{t}_1, \tag{57}$$

$$\mathbf{H}_1\cdot\mathbf{t}_2=\mathbf{H}_2\cdot\mathbf{t}_2+\mathbf{H}_i\cdot\mathbf{t}_2. \tag{58}$$

In general, we have the remaining two equations arising from the continuity of the normal components of $\mathbf{D}=\epsilon\mathbf{E}$ and $\mathbf{B}=\mu\mathbf{H}$. Here $\mathbf{n}$ (without the subscripts) refers to the normal vector and not one of the field functions :

$$\epsilon_1\mathbf{E}_1\cdot\mathbf{n}=\epsilon_2\mathbf{E}_2\cdot\mathbf{n}+\epsilon_2\mathbf{E}_i\cdot\mathbf{n}, \tag{59}$$

$$\mu_1\mathbf{H}_1\cdot\mathbf{n}=\mu_2\mathbf{H}_2\cdot\mathbf{n}+\mu_2\mathbf{H}_i\cdot\mathbf{n}. \tag{60}$$

Each of the terms above of the form $\mathbf{E}\cdot\mathbf{t}$ is expressible as an infinite series when we substitute the expression for the appropriate field expansion. Thus, for example,

$$\mathbf{E}_1\cdot\mathbf{t}_1=\sum_{n=1}^{\infty}\sum_{m=0}^{n}\{a_{mn}^{1j}(\mathbf{m}_{emn}^{1j}\cdot\mathbf{t}_1)+b_{mn}^{1j}(\mathbf{n}_{omn}^{1j}\cdot\mathbf{t}_1)\}. \tag{61}$$

In order to obtain these coefficients we approximate the field expansions by truncating the series to a finite number of terms. The assumption of azimuthal symmetry allows the different m values to decouple [25].

The remaining procedure is fairly straightforward. We solve a linear system of equations derived from the boundary condition equations. The incident field (source) terms are completely known and they constitute the input vector. The left hand sides are linear expressions in the unknowns a_{mn}^{1j}, b_{mn}^{1j}, a_{mn}^{2h}, and b_{mn}^{2h} (with $n\in m,m+1,\ldots,m+N-1$). Specifying more points on the boundary or including more terms in the truncated expansion for the scattered fields would tend to give us increasingly better approximations to the exact solution but at the expense of increasing the size of the boundary-condition matrix. The system is deliberately allowed to be an overdetermined system (number of rows $M>N$, number of columns) since we are considering scattering objects with arbitrary shapes. We can solve the system in a least-squares sense.

In Fig. 4 we show the near field calculation of a spherical scatterer $(L=0)$, with the incident wave approaching along $\theta=0^{\mathbf{o}}$. The sphere allows us to verify the method of solution. Poynting vector calculations done using the basis function solution, both in the near and far fields [25], are in excellent agreement with existing literature [26]. In Fig. 5 we show the calculation done on a capsule shaped scatterer with the incident wave approaching along $\theta=60^{\mathbf{o}}$. As discussed before, the scattering object has azimuthal symmetry about the z axis. However, in the latter case, the incidence direction of the incident plane wave breaks the symmetry. The continuity of the field patterns at the boundary validates the field calculations. Previously, such calculations were possible using numerical methods only.

VII. CONCLUSION

An exact method of solving for electromagnetic scattering has been developed. We have demonstrated the existence of a complete set of vector eigenfunctions for problems in electromagnetic scattering. The eigenfunction expansion converges sufficiently rapidly to be of importance for numerical computations. The expansion coefficients are determinable by imposing the boundary conditions. The solution method has been developed for the spherical-polar coordinate system and is therefore applicable for scattering objects whose boundaries do not extend to infinity. However, the technique is sufficiently general and can be generalized to other coordinate systems.

[1] D. S. Jones, *Methods in Electromagnetic Wave Propagation*, 2nd ed. (Clarendon Press, Oxford, UK, 1994).
[2] C. Muller, *Foundations of the Mathematical Theory of Electromagnetic Waves* (Springer-Verlag, Berlin, 1969).
[3] J. J. Bowman, T. B. A. Senior, and P. L. E. Uslenghi, *Electromagnetic and Acoustic Scattering by Simple Shapes* (Hemisphere Publishing Company, New York, 1987).
[4] P. K. Aravind, R. W. Rendell, and H. Metiu, Chem. Phys. Lett. **85**, 396 (1982).
[5] P. K. Aravind and H. Metiu, Surf. Sci. **124**, 506 (1983).
[6] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1975).
[7] M. Born and E. Wolf, *Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light* (Pergamon Press, Oxford, UK, 1980).
[8] H. N. Kritikos and D. L. Jaggard, *Recent Advances in Electromagnetic Theory* (Springer-Verlag, New York, 1990).
[9] W. W. Hansen, Physics **7**, 460 (1936).
[10] W. W. Hansen, J. Appl. Phys. **8**, 282 (1937).
[11] W. W. Hansen, Phys. Rev. **47**, 139 (1935).
[12] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
[13] A. L. Aden and M. Kerker, J. Appl. Phys. **22**, 1242 (1951).
[14] H. C. van de Hulst, *Light Scattering by Small Particles* (Dover, New York, 1981).
[15] C. F. Bohren and D. R. Huffman, *Absorption and Scattering of Light by Small Particles* (John Wiley & Sons, New York, 1983).
[16] J. R. Wait, Can. J. Phys. **33**, 189 (1955).
[17] K. M. Siegel, J. W. Crispin, and C. E. Schensted, J. Appl. Phys. **26**, 309 (1955).
[18] C. E. Schensted, J. Appl. Phys. **26**, 306 (1955).
[19] P. M. Morse and H. Feshback, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vols. 1 and 2.
[20] G. Arfken, *Mathematical Methods for Physicists*, 2nd ed. (Academic, New York, 1970).
[21] H. W. Wyld, *Mathematical Methods for Physics* (W. A. Benjamin, Inc., Ontario, 1976).
[22] G. F. Simmons, *Introduction to Topology and Modern Analysis* (McGraw-Hill, Tokyo, 1963).
[23] P. R. Halmos, *Naive Set Theory* (D. Van Nostrand Company, Inc., New York, 1960).
[24] T. M. Apostol, *Mathematical Analysis; A Modern Approach to Advanced Calculus* (Addison-Wesley, Reading, MA, 1957).
[25] D. Sarkar, Ph.D. thesis, Rice University, Houston, 1996 (unpublished).
[26] U. Kreibig and M. Vollmer, *Optical Properties of Metal Clusters* (Springer-Verlag, Berlin, 1995).